Lecture 18

Wednesday, October 20, 2021 9:43 AM

Last Time: Fubini's Theorem

Ex. Compute $\iint_R y e^{-xy}\, dA$ on $R = [0,2] \times [0,3]$

Sol: $\iint_R y e^{-xy}\, dA = \int_{x=0}^{2} \int_{y=0}^{3} y e^{-xy}\, dy\, dx$

Compute inner integral

Compute outer integral

$\int_0^3 y e^{-xy}\, dy$

$u = y \quad dv = e^{-xy} dy$

$du = dy \quad v = -\frac{1}{x} e^{-xy}$

$-\frac{y}{x} e^{-xy} - \int -\frac{1}{x} e^{-xy}\, dy$

$-\frac{y}{x} e^{-xy} - \frac{1}{x^2} e^{-xy} \Big/_0^3$

$\left(-\frac{3}{x} e^{-3x} - \frac{1}{x^2} e^{-3x}\right) - \left(0 - \frac{1}{x^2}\right)$

$e^{-3x}\left(-\frac{3}{x} - \frac{1}{x^2}\right) + \frac{1}{x^2}$

$\int_0^2 e^{-3x}\left(-\frac{3}{x} - \frac{1}{x^2}\right) + \frac{1}{x^2}\, dx$ ← outer integral

This does not work

Retry with y on outside x on inside

$\int_{y=0}^{3} \int_{x=0}^{2} y e^{-xy}\, dx\, dy$

inner first

outer last

$\int_0^2 y e^{-xy}\, dx = -e^{-xy} \Big/_0^2 = -e^{-2y} - \left(e^{-0y}\right)$

$= -e^{-2y} + 1$

$$= -e^{-2y}+1$$

$$\int_0^3 -e^{-2y}+1\, d_y = y + \tfrac{1}{2}e^{-2y}\Big/_0^3$$

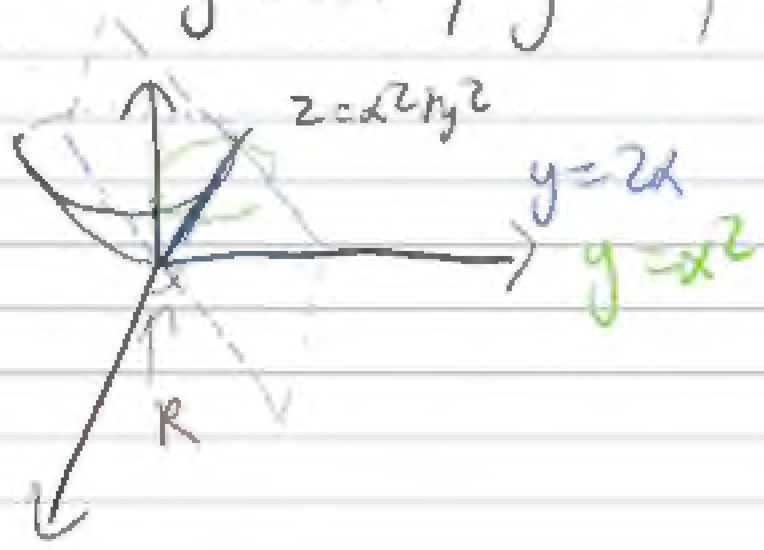
$$\left(3+\tfrac{1}{2}e^{-6}\right) - \left(0+\tfrac{1}{2}e^{0}\right) = \boxed{\tfrac{5}{2}+\tfrac{1}{2}e^{-6}}$$

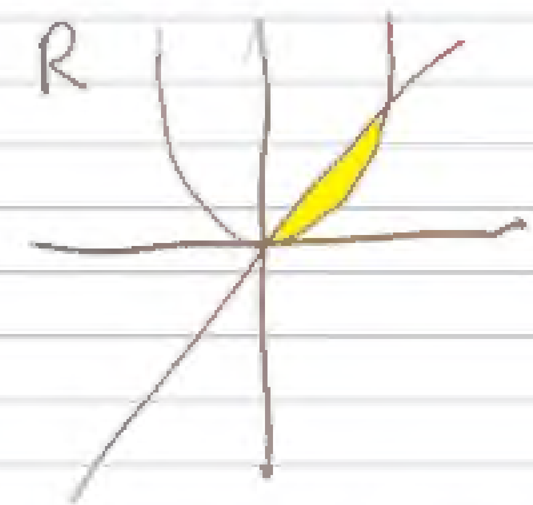
Goal: Integrate over a more complicated Region

Ex Compute net volume of solid bound by

$z = x^2+y^2 \qquad y = 2x, \; y = x^2, \; z = 0$

Picture:





R $\{(x,y) \mid (x,y)$ is between $y=2x$ and $y=x^2$

$(x,y) \quad x^2 \le y \le 2x$

$$\text{Vol}(S) = \iint_R \left((x^2+y^2) - 0\right)\, dA$$

$$\int_{x=0}^{2}\int_{y=x^2}^{2x} (x^2+y^2)\, dy\, dx$$

$x^2 = 2x$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0, \; x = 2$

$$\int_{x^2}^{2x} x^2 + y^2 \, dy$$

$$yx^2 + \frac{y^3}{3}\Big/_{x^2}^{2x}$$

$$\int_{x=0}^{2}\left(2x^3 + \frac{(2x)^3}{3} - x^4 + \frac{(x^4)^3}{3}\right)$$

$$\int_0^2 \left(\frac{14}{3}x^3 - x^4 \cdot \frac{1}{3}x^6\right) dx$$

$$\frac{14}{12}x^4 - \frac{1}{5}x^5 - \frac{1}{21}x^7 \Big/_0^2$$

$$\frac{14}{12}\cdot 2^{4} - \frac{1}{5}\cdot 2^5 - \frac{1}{21}\cdot 2^7 - 0$$

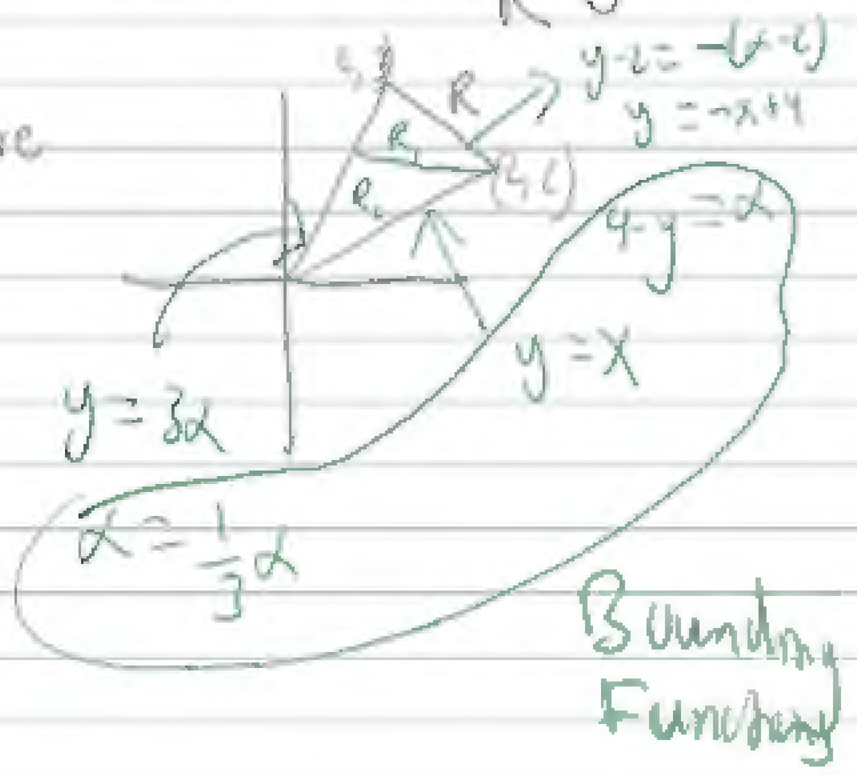
$$\frac{14}{12}\cdot 2^4 - \frac{1}{5}\cdot 2^5 - \frac{1}{21}\cdot 2^7 - 0$$

$$\frac{14}{\cancel{12}_{3}}\cdot \cancel{16}^{4} - \frac{1}{5}\cdot 32 - \frac{1}{21}\cdot 128$$

$$\boxed{\frac{56}{3} - \frac{32}{5} - \frac{128}{21}}$$

Ex. Compute $\iint_R y\, dA$ over R the triangle w/ vertices $(0,0), (1,3), (2,2)$

Picture



$R = \{(x,y) : 0 \le y \le 3$

$R = R_1 \cup R_2$

$R_1 = \{(x,y) : 2 \le y \le 3,\ \frac{1}{3}y \le x \le 4-y\}$

$R_2 = \{(x,y) : 0 \le y \le 2,\ \frac{1}{3}y \le x \le y\}$

$$\iint_R y\, dA = \iint_{R_1} y\, dA + \iint_{R_2} y\, dA$$

$$\iint_{R_1} y\, dA = \int_{y=2}^{3}\int_{x=\frac{1}{3}y}^{4-y} y\, dx\, dy$$

$$\int_{\frac{1}{3}y}^{4-y} y\, dx$$

$$xy \Big|_{\frac{1}{3}y}^{4-y}$$

$$\int_{y=2}^{3}\left((4-y)y - \tfrac{1}{3}y\cdot y\right) dy$$

$$y\left(4-y-\tfrac{1}{3}y\right)$$

$$\iint_{R_2} y\, dA = \int_{y=0}^{2}\int_{x=\frac{1}{3}y}^{y} y\, dx\, dy$$

$$\int_{x=\frac{1}{3}y}^{y} y\, dx$$

$$xy \Big|_{\frac{1}{3}y}^{y}$$

$$\int_{y=0}^{2}\left(y^2 - \tfrac{1}{3}y^2\right) dy$$

$$\int_{y=0}^{2} y^2\left(1-\tfrac{1}{3}\right) dy$$

$y(4-y-\frac{1}{3}y)$

$4y-y^2-\frac{1}{3}y^2$

$\int_{y=2}^{3}\left(4y-\frac{4}{3}y^2\right)dy$

$2y^2-\frac{4}{9}\Big/_{2}^{3}$

$R_1 = \frac{14}{9}$

$\int_{y=0} \ldots \left(\ldots \frac{}{3}\right)dy$

$\frac{2}{3}\cdot\frac{1}{3}y^3\Big/_{0}^{2}$

$\frac{2}{9}(8-0)$

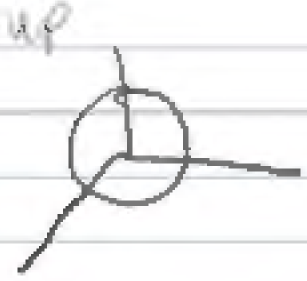
$R_2 = \frac{16}{9}$

$\iint_R y\,dA = \frac{14}{9}+\frac{16}{9}$

$= \boxed{\frac{10}{3}}$

Motivating Question: What is the volume of a sphere?

Setup



$S=\{(x,y,z): x^2+y^2+z^2=r^2\}$

for $(x,y)\in \mathbb{R}$

$z=\pm\sqrt{r^2-x^2-y^2}$

$\text{Vol}(S)=\iint_R 2\sqrt{r^2-x^2-y^2}\,dA$